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A MIXTURE THEORY WITH A DIRECTOR FOR LINEAR ELASTODYNAMICS OF PERIODICALLY LAMINATED MEDIA†

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Abstract—The asymptotic method of multiple scales is used to construct a continuum theory with microstructure for the linear elastodynamics of a periodically laminated medium. The resulting theory is in the form of a homogeneous binary mixture theory of micromorphic materials with a common director oriented normal to the interfaces. The model contains nine conservation equations—six for the linear momenta of both constituents and three for the director momentum. The asymptotically derived constitutive equations contain mixture properties which, in contrast to phenomenological theories, are determined solely from the properties of the individual constituents and their volume fractions. The mixture conservation and constitutive equations are complemented by an appropriate set of boundary conditions determined by a variational procedure.

The efficacy of the model is assessed by comparison of predicted and exact phase velocity spectra for waves propagating at oblique incidence to the layers. The excellent agreement observed indicates that the model is useful for studying the dynamic behavior of laminated composites. Further, the method of multiple scales appears to provide an effective approach to the accurate determination of the large scale behavior of a material which exhibits small scale periodic heterogeneity.

INTRODUCTION

Scope

In this paper an asymptotic method is used to develop a three-dimensional continuum theory with microstructure for the linear elastodynamics of a periodically laminated composite. The model, which combines the desirable features of several existing theories[1, 2], has the form of a homogeneous binary mixture theory of micromorphic materials with a common director.

Model construction is based upon the observation that, along a direction normal to the laminae, there are two length scales over which significant variations in displacement and stress profiles occur and that these scales differ by at least an order of magnitude in most problems of practical interest. This observation suggests the use of the asymptotic method of multiple scales[3, 23] to transform the original three-dimensional problem into one with four independent variables, the new variable being a microcoordinate along the direction normal to the layers. Since the material properties then become functions of the microcoordinate only and are periodic, the transformed problem takes the form of a "generalized wave guide"-type problem which is easily amenable to solution via an asymptotic procedure which is developed here as an extension of the regular asymptotic method used by Hegemier[4–6] for construction of continuum models of wave guide-type propagation in composite materials.

Subsequent to the derivation of the differential equations of the mixture theory, a functional is constructed, the extremization of which yields the mixture equations as the Euler equations. The resulting variational principle furnishes appropriate boundary data for the mixture theory. Finally, the efficacy of the proposed theory is tested by comparison of predicted vs exact phase velocity spectra for time harmonic waves propagating at oblique incidence to the laminae interfaces.

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Relation to other theories

Although the literature on the mechanics of materials with microstructure is too large to be reviewed here, it is appropriate to refer to some investigations which are perhaps typical. In probably the earliest attempt to model materials with internal structure, the concept of directors was introduced by the Cosserat brothers[7]. The concept was subsequently generalized by Ericksen and Truesdell[8]. Although it was introduced much earlier, the mixture theoretic formalism for elastodynamic behavior of multi-component materials was spotlighted by the works of Green and Naghdi[9], Bowen[10] *et al.* As a natural extension of these two concepts, Allen and Kline[11] and Twiss and Eringen[12] explored mixtures of materials each of which was assigned a separate director. It is emphasized that all these studies were based upon the macroscopic, phenomenological point of view wherein little information is assumed or used concerning the material microstructure. Although such investigations eminently serve the purpose of elucidating the qualitative behavior to be expected of materials with microstructure, their utility for composite materials is severely limited by the fact that they necessitate the performance of (considerable numbers of) experiments for the determination of the constants appearing in the constitutive equations. This is the case even when the geometries and properties of the composite constituents are well-defined.

Evidently, the foregoing limitation can be overcome only if construction of a macroscopic theory is based upon micromechanical considerations. Examples of such approaches can be found in the works of Bedford and Stern[13], Sun *et al.*[1] and Hegemier *et al.*[14]. Of course, the technique described by Hegemier[5] and used in a number of studies of wave guide-type problems[4, 6, 15] was developed with the explicit objective of including microstructural details in the macroscopic model of composite materials. As was noted previously, this paper describes an extension of this procedure to problems which are not of the wave guide-type. Thus, the method presented herein can be viewed as a special procedure for solution of a restricted class of homogenization problems[16, 17].

FORMULATION

Consider a domain $\bar{D}$ which contains a large number of (two) alternating linearly elastic homogeneous laminae with perfect bonds as illustrated in Fig. 1. A typical cell, which represents the geometrical microstructure of the composite, is shown in Fig. 2. Let a rectangular Cartesian reference system $\bar{x}_1, \bar{x}_2, \bar{x}_3$ be selected with $\bar{x}_3$ normal to the middle plane of the central cell lamina and $\bar{x}_2, \bar{x}_3$ in this plane.

For notational convenience forms $(\ )^{(\alpha)}$, $\alpha = 1, 2$ will denote quantities associated with material α. The usual Cartesian indicial notation will be employed where Latin indicies range from 1 to 3 and repeated indices imply the summation convention, unless otherwise noted. In

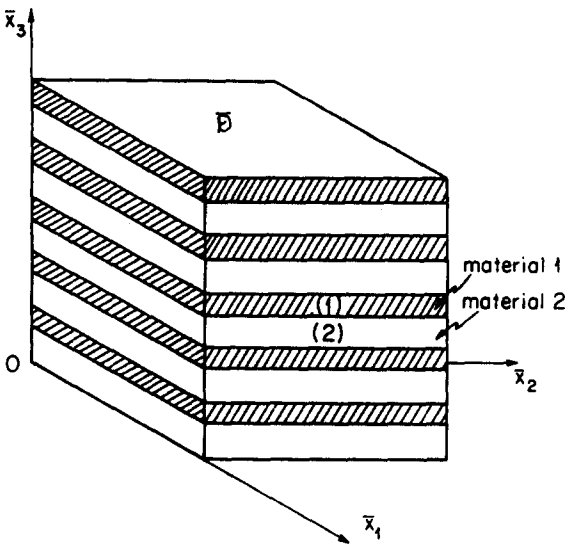

Fig. 1. Geometry and coordinate system.

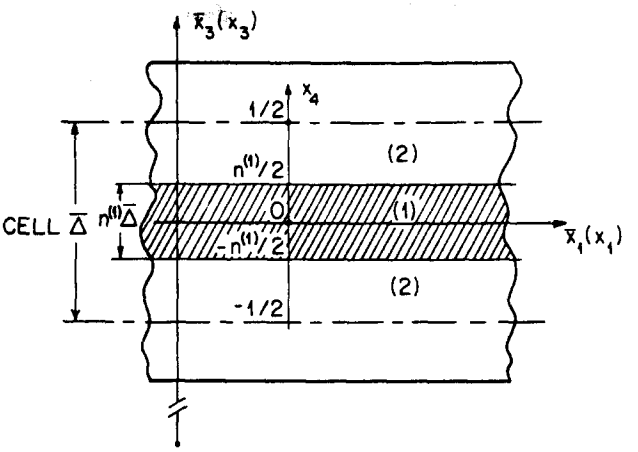

Fig. 2. Unit cell.

addition, the notations $(\ )_{,i} \equiv \partial(\ )/\partial\bar{x}_i$ and $(\dot{}) \equiv \partial(\ )/\partial\bar{t}$ will be employed where $\bar{t}$ represents time. Quantities of the form $(\bar{})$ and $(\ )$ denote dimensional and nondimensional variables, respectively.

With the aid of the foregoing notation, the governing relations for the displacement vector $\bar{u}_i^{(\alpha)}$ and the stress tensor $\bar{\sigma}_{ij}^{(\alpha)}$ in the two constituents are:

(a) *Equations of motion*:

$$\bar{\sigma}_{ji,j}^{(\alpha)} = \bar{\rho}^{(\alpha)}\ddot{\bar{u}}_i^{(\alpha)}, \quad \bar{\sigma}_{ij}^{(\alpha)} = \bar{\sigma}_{ji}^{(\alpha)}, \tag{1}$$

where $\bar{\rho}^{(\alpha)}$ is the mass density;

(b) *Constitutive relations*:

$$\bar{\sigma}_{ij}^{(\alpha)} = \bar{\lambda}^{(\alpha)}\bar{u}_{k,k}^{(\alpha)}\delta_{ij} + \bar{\mu}^{(\alpha)}(\bar{u}_{i,j}^{(\alpha)} + \bar{u}_{j,i}^{(\alpha)}), \tag{2}$$

where $\bar{\lambda}^{(\alpha)}$, $\bar{\mu}^{(\alpha)}$ are Lamé's constants and δ_{ij} is the Kronecker delta;

(c) *Interface continuity conditions*:

$$\bar{u}_i^{(1)} = \bar{u}_i^{(2)}, \quad \bar{\sigma}_{3i}^{(1)} = \bar{\sigma}_{3i}^{(2)} \quad \text{on interfaces.} \tag{3}$$

For complete specification of an initial boundary value problem, one must also specify:

(d) *Initial conditions at* $\bar{t} = 0$ *and appropriate boundary data along the boundary of* $\bar{D}$.

Conditions (a)–(d) define a well-posed initial boundary value problem. The direct solution of this problem is, however, extremely difficult due to the large number of interfaces in most cases of practical interest. The objective of the subsequent analysis is to alleviate such difficulties by deriving a set of partial differential equations with constant coefficients whose solution can be utilized to approximate the solution of the problem posed by (a)–(d). This effort is facilitated by a judicious scaling of both dependent and independent variables. To this end the following quantities are introduced:

$\bar{\Lambda}$ typical macrosignal wavelength or macrodimension;
$\bar{\Delta}$ composite cell length;
$\bar{c}_{(m)}, \bar{\rho}_{(m)}$ reference wave velocity and macrodensity;
$\bar{E}_{(m)} = \bar{\rho}_{(m)}\bar{c}_{(m)}^2$ reference composite elastic modulus;
$\bar{t}_{(m)} = \bar{\Lambda}/\bar{c}_{(m)}$ typical macrosignal travel time;
$\epsilon = \Delta/\bar{\Lambda}$ ratio of micro-to-macrodimensions.

With the aid of the foregoing notation, the following nondimensional quantities are now defined:

$$\begin{gathered}(x_1, x_2, x_3) = (\bar{x}_1, \bar{x}_2, \bar{x}_3)/\bar{\Lambda}, \quad t = \bar{t}/\bar{t}_{(m)}, \\ u_i^{(\alpha)} = \bar{u}_i^{(\alpha)}/\bar{\Lambda}, \quad \sigma_{ij}^{(\alpha)} = \bar{\sigma}_{ij}^{(\alpha)}/\bar{E}_{(m)}, \quad i, j = 1-3, \\ \rho^{(\alpha)} = \bar{\rho}^{(\alpha)}/\bar{\rho}_{(m)}, \quad (\lambda, \mu)^{(\alpha)} = (\bar{\lambda}, \bar{\mu})^{(\alpha)}/\bar{E}_{(m)}.\end{gathered} \tag{4}$$

With the variables defined according to (4), the material properties are seen to be periodic in the x_3-direction with period ϵ. Thus, the field variables are expected to vary significantly over two length scales along the x_3-direction. One such scale is, of course, of the order of the signal wavelength which is unity in the scaled system. The other length scale is of the order of the cell dimension ϵ. This suggests a two-variable expansion procedure[3]. Therefore, new independent variables x_4, x_3^* are introduced according to

$$\begin{aligned} x_4 &= \phi^{-1}(\epsilon)x_3, \quad \phi(\epsilon)\to 0 \quad \text{as} \quad \epsilon\to 0, \\ x_3^* &= \psi(\epsilon)x_3, \quad \psi(\epsilon)\to 1 \quad \text{as} \quad \epsilon\to 0. \end{aligned} \tag{5a}$$

For purposes of the present analysis it will suffice to select

$$\phi(\epsilon)=\epsilon, \quad \psi(\epsilon)=1. \tag{5b}$$

All field variables are now functions of x_4 as well as the macrovariables x_i (i = 1–3), i.e.

$$f(x_1, x_2, x_3, t; \epsilon) = f^*(x_1, x_2, x_3, x_4, t; \epsilon). \tag{6}$$

For any field variable (6), one obtains

$$\frac{\partial}{\partial x_3} f(x_1, x_2, x_3, t; \epsilon) = \frac{\partial}{\partial x_3} f^*(x_1, x_2, x_3, x_4, t; \epsilon) + \frac{1}{\epsilon}\frac{\partial}{\partial x_4} f^*(x_1, x_2, x_3, x_4, t; \epsilon). \tag{7}$$

In what follows the superscript * will be dropped for simplicity.

On introduction of x_i, x_4, and the nondimensional variables (4), eqns (1)–(3) take the form

$$\sigma_{ji,j}^{(\alpha)} + \frac{1}{\epsilon}\sigma_{3i,4}^{(\alpha)} = \rho^{(\alpha)}\ddot{u}_i^{(\alpha)}, \quad \sigma_{ij}^{(\alpha)} = \sigma_{ji}^{(\alpha)}, \tag{8}$$

$$\sigma_{ij}^{(\alpha)} = \delta_{ij}\lambda^{(\alpha)}u_{k,k}^{(\alpha)} + \mu^{(\alpha)}(u_{i,j}^{(\alpha)} + u_{j,i}^{(\alpha)}) + \frac{1}{\epsilon}\{\lambda^{(\alpha)}u_{3,4}^{(\alpha)}\delta_{ij} + \mu^{(\alpha)}(\delta_{j3}u_{i,4}^{(\alpha)} + \delta_{i3}u_{j,4}^{(\alpha)})\}, \tag{9}$$

$$u_i^{(1)} = u_i^{(2)}, \quad \sigma_{3i}^{(1)} = \sigma_{3i}^{(2)} \quad \text{on interfaces.} \tag{10}$$

For the sake of notational convenience, the dependence of the field variables on all four independent variables x_i, x_4 has not been explicitly stated in (8)–(10) and the notation $(\,)_{,i} \equiv \partial(\,)/\partial x_i$, $\dot{(\,)} \equiv \partial(\,)/\partial t$ has been utilized.

Equations (8)–(10) can be viewed as a set of partial differential equations in x_i, x_4 and t, in which the coefficients are independent of x_i, t, but are periodic functions of x_4 with a unit period. If the domain of definition of (8)–(10) was such that $-\infty < x_4 < \infty$, and if the initial conditions were quiescent or periodic in x_4, it would follow from Floquet Theory[19, 20] that there exists a set of solutions of (8)–(10) which are periodic in x_4 with unit period length. This periodicity condition is expected to continue to hold, at least in the interior of the domain in which the solution is sought, so long as the length of the domain in the x_3-direction is sufficiently large. Consequently, eqns (8)–(10) will be analyzed in what follows based upon the premise that all field variables are periodic in the microcoordinate x_4, with the understanding that such an analysis may be valid only at points sufficiently far removed from the boundaries of the domain in which the solution is to be obtained.

As a result of the aforementioned periodicity condition, it is necessary to treat only a typical unit cell associated with the x_4-direction. With no loss in generality (the particular choice of the unit cell is immaterial so long as the periodicity condition is imposed) the cell may, for convenience, be defined by

$$A^{(1)} \equiv \left[x_4 \middle| |x_4| < \frac{n^{(1)}}{2}\right], \quad A^{(2)} \equiv \left[x_4 \middle| \frac{n^{(1)}}{2} < |x_4| < \frac{1}{2}\right] \tag{11}$$

In (11), $A^{(\alpha)}$ denotes the domain occupied by material α in the cell and $n^{(\alpha)}$ is the volume

fraction of material α; the latter satisfy

$$n^{(1)} + n^{(2)} = 1. \tag{12}$$

With the above choice of the unit cell, the interface condition (10) must be satisfied at $|x_4| = n^{(1)}/2$ and the periodicity condition is given by

$$f(x_i, -1/2, t; \epsilon) = f(x_i, 1/2, t; \epsilon), \tag{13}$$

where f represents any field variable.

The formulation of the problem is now complete. The key elements introduced thus far are (1) the microcoordinate x_4 and (2) the periodicity of all field variables in this microcoordinate. It will be demonstrated in the following analysis that the introduction of the additional independent variable in fact simplifies the problem considerably since it facilitates the determination of the field-quantity variations within the unit cell and allows eventual elimination of the microcoordinate by a suitable averaging process defined over a unit cell.

It should be noted here that eqns (8)–(10) as well as the periodicity condition can be derived by using an alternate definition of x_4 based upon the concept of a movable origin which was introduced by Burgers[16]. Thus, formally, the results presented below can also be derived on the basis of the concept of a moveable origin.

MIXTURE EQUATIONS

Binary mixture equations for the composite can be immediately obtained by introducing an averaging operation according to

$$f^{(\alpha a)}(x_i, t; \epsilon) = \frac{1}{n^{(\alpha)}} \int_{A^{(\alpha)}} f^{(\alpha)}(x_i, x_4, t; \epsilon)\, dx_4. \tag{14}$$

Upon averaging (8) with use of the periodicity condition for the stress components σ_{3i}, one obtains the momentum equations

$$\sigma^{(\alpha p)}_{ji,j} - \rho^{(\alpha p)} \ddot{u}_i^{(\alpha a)} = (-1)^{\alpha} P_i, \tag{15}$$

where P_i denotes the components of an interaction (body) force vector per unit volume, given by

$$P_i \equiv \frac{1}{\epsilon} \{\sigma^{(\alpha)}_{3i}|_{x_4 = n^{(1)}/2} - \sigma^{(\alpha)}_{3i}|_{x_4 = -n^{(1)}/2}\}. \tag{16}$$

In (15), the definitions for "partial" quantities,

$$\sigma^{(\alpha p)}_{ij} \equiv n^{(\alpha)} \sigma^{(\alpha a)}_{ij}, \quad \rho^{(\alpha p)} \equiv n^{(\alpha)} \rho^{(\alpha)} \tag{17}$$

have also been utilized.

The momentum eqns (15) are complemented by mixture constitutive relations obtained by averaging (9) and applying the periodicity condition; this yields

$$\sigma^{(\alpha p)}_{ij} = [\lambda^{(\alpha p)} u^{(\alpha a)}_{k,k} \delta_{ij} + \mu^{(\alpha p)}(u^{(\alpha a)}_{i,j} + u^{(\alpha a)}_{j,i})] + (-1)^{\alpha+1}[\lambda^{(\alpha)} S_{33} \delta_{ij} + \mu^{(\alpha)}(\delta_{j3} S_{3i} + \delta_{i3} S_{3j})], \tag{18a}$$

where

$$\lambda^{(\alpha p)} \equiv n^{(\alpha)} \lambda^{(\alpha)}, \quad \mu^{(\alpha p)} \equiv n^{(\alpha)} \mu^{(\alpha)} \tag{18b}$$

and

$$S_{3i} \equiv \frac{1}{\epsilon} \{u_i^{(\alpha)}|_{x_4 = n^{(1)}/2} - u_i^{(\alpha)}|_{x_4 = -n^{(1)}/2}\}. \tag{18c}$$

The microcoordinate-independent quantities S_{3i} $(i = 1, 2, 3)$, which appear naturally in the constitutive equations as a consequence of the averaging process utilized, shall be termed director displacements associated with a director (vector) oriented along the x_3-axis. In the subsequent analysis, the director displacements will be treated as dependent variables in addition to the average displacements $u_i^{(\alpha\alpha)}$.

In order to "close" the mixture relations, it is necessary to construct (a) appropriate conservation equations associated with director displacements and (b) constitutive equations for P_i and for the stress-type quantities associated with the director displacements. For this purpose an asymptotic procedure will be utilized which is an extension of the regular asymptotic method described by Hegemier[5].

ASYMPTOTIC ANALYSIS

The premise that the composite macrodimension is much larger than the microdimension, i.e. $\epsilon \ll 1$, and the form of the scaled eqns (8)–(10) suggest the expansion of the dependent variables in the asymptotic series

$$\{u_i^{(\alpha)}, \sigma_{ij}^{(\alpha)}\}(x_k, x_4, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^n \{u_{i(n)}^{(\alpha)}, \sigma_{ij(n)}^{(\alpha)}\}(x_k, x_4, t). \tag{19}$$

If (19) is substituted into (8)–(10) and the coefficients of different powers of ϵ are equated to zero, a sequence of problems is obtained. The first of the equations in this sequence furnishes

$$\sigma_{3i(0),4}^{(\alpha)} = u_{i(0),4}^{(\alpha)} = 0. \tag{20}$$

Thus the quantities $\sigma_{3i(0)}^{(\alpha)}$, $u_{i(0)}^{(\alpha)}$ are independent of the microcoordinate x_4, i.e.

$$\{u_i^{(\alpha)}, \sigma_{3i}^{(\alpha)}\}_{(0)} = \{u_i, \sigma_{3i}\}_{(0)}(x_1, x_2, x_3, t) \tag{21}$$

where the zeroth order expansions of the interface conditions (10) have also been utilized. The remaining system of equations obtained from (8) to (10) and (19) are

$$\sigma_{ji(n),j}^{(\alpha)} + \sigma_{3i(n+1),4}^{(\alpha)} = \rho^{(\alpha)} \ddot{u}_{i(n)}^{(\alpha)}, \quad \sigma_{ji(n)}^{(\alpha)} = \sigma_{ij(n)}^{(\alpha)}, \tag{22a}$$

$$\sigma_{ij(n)}^{(\alpha)} = \delta_{ij}\lambda^{(\alpha)}(u_{k(n),k}^{(\alpha)} + u_{3(n+1),4}^{(\alpha)}) + \mu^{(\alpha)}(u_{i(n),j}^{(\alpha)} + u_{j(n),i}^{(\alpha)} + \delta_{j3}u_{i(n+1),4}^{(\alpha)} + \delta_{i3}u_{j(n+1),4}^{(\alpha)}). \tag{22b}$$

Because of (21), the higher order terms in (19) must satisfy a normalization condition; a convenient choice is

$$u_{i(n)}^{(1)} = \sigma_{3i(n)}^{(1)} = 0 \quad \text{at} \quad x_4 = 0, \quad n \geq 1. \tag{23}$$

From (21), (22) one concludes that, for $n \geq 1$,

$$u_{i(2n)}^{(\alpha)} \quad \text{and} \quad \sigma_{3i(2n)}^{(\alpha)} \quad \text{are even functions of } x_4, \tag{24a}$$

$$u_{i(2n-1)}^{(\alpha)} \quad \text{and} \quad \sigma_{3i(2n-1)}^{(\alpha)} \quad \text{are odd functions of } x_4. \tag{24b}$$

These properties follow from the choice of the unit cell, i.e. from the symmetrical distribution of material properties within it. In view of (24) and the periodicity condition, one also obtains

$$u_{i(2n-1)}^{(2)} = \sigma_{3i(2n-1)}^{(2)} = 0 \quad \text{on} \quad x_4 = \pm\frac{1}{2}, \quad n \geq 1. \tag{25}$$

To be added to the above are the interface conditions for continuity of displacement and traction vectors on $x_4 = \pm n^{(1)}/2$. These are obtained from (10) and have the following form:

$$u_{i(n)}^{(1)} = u_{i(n)}^{(2)}, \quad \sigma_{3i(n)}^{(1)} = \sigma_{3i(n)}^{(2)} \quad \text{on} \quad x_4 = \pm n^{(1)}/2. \tag{26}$$

FIRST ORDER DISPLACEMENT FIELD AND CONSERVATION OF DIRECTOR MOMENTUM

Equations (20) and (22b), with $n = 0$, furnish

$$u^{(\alpha)}_{i(1),4} = B^{(\alpha)}(x_1, x_2, x_3, t). \tag{27}$$

The function $B^{(\alpha)}$ is related to the director displacement by use of (18c) with the result†

$$u^{(\alpha)}_{i(1)} = S_{3i}(x_i, t) g^{(\alpha)}(x_4) \tag{28}$$

where

$$g^{(1)}(x_4) = \frac{x_4}{n^{(1)}}, \qquad g^{(2)}(x_4) = \frac{1}{n^{(2)}} \left[\frac{|x_4|}{2x_4} - x_4 \right]. \tag{29}$$

Equations (23, 25, 26) were also employed in the derivation of (28, 29).

The introduction of the dependent variables S_{3i} necessitates additional conservation relations. These are obtained by taking g-moments of (22a) with $n = 1$ and

$$g \equiv \begin{cases} g^{(1)}(x_4) & \text{on} \quad A^{(1)} \\ g^{(2)}(x_4) & \text{on} \quad A^{(2)}. \end{cases} \tag{30}$$

Thus, from (29, 30, 22a) one obtains

$$M_{3ji,j} + R_{3i} = \rho_{(m)} \ddot{S}_{3i}/12, \tag{31}$$

where

$$M_{3ji} \equiv \sum_{\alpha=1}^{2} M^{(\alpha)}_{3ji} \equiv \sum_{\alpha=1}^{2} \int_{A^\alpha} g^{(\alpha)}(x_4) \sigma^{(\alpha)}_{ij(1)} \, dx_4, \tag{32a}$$

$$R_{3i} \equiv \sum_{\alpha=1}^{2} R^{(\alpha)}_{3i} \equiv \sum_{\alpha=1}^{2} \int_{A^{(\alpha)}} g^{(\alpha)}(x_4) \sigma^{(\alpha)}_{3i(2),4} \, dx_4. \tag{32b}$$

$$\rho_{(m)} \equiv \rho^{(1p)} + \rho^{(2p)}. \tag{33}$$

The conservation equations represented by (31) contain the additional variables $M^{(\alpha)}_{3ji}$ and $R^{(\alpha)}_{3i}$. The constitutive relations for these quantities are developed in the following together with those for P_i.

MIXTURE THEORY BASED ON TWO LOWEST ORDER SYSTEMS

Constitutive relations for interaction forces P_i

Equations (22a) for $n = 0$, together with (21), yields

$$\sigma^{(\alpha)}_{3i(1),4} = C^{(\alpha)}_i(x_k, t). \tag{34}$$

If (34) is now integrated over $A^{(\alpha)}$ and use is made of (16), (19), (24), (25) one obtains

$$C^{(\alpha)}_i(x_k, t) = (-1)^{\alpha+1} P_i / n^{(\alpha)} \tag{35}$$

whence

$$\sigma^{(\alpha)}_{3i(1),4} = (-1)^{\alpha+1} P_i / n^{(\alpha)}. \tag{36}$$

Moreover, the constitutive eqn (22b) together with (28) furnishes

$$\sigma^{(\alpha)}_{3j(1)} = \delta_{3j} \lambda^{(\alpha)} (S_{3k,k} g^{(\alpha)}(x_4) + u^{(\alpha)}_{3(2),4}) + \mu^{(\alpha)} [(S_{33,j} + S_{3j,3}) g^{(\alpha)}(x_4) + u^{(\alpha)}_{j(2),4} + \delta_{j3} u^{(\alpha)}_{3(2),4}]. \tag{37}$$

†For this calculation $u^{(\alpha)}_i$ were represented in (18c) by terms up to and including $O(\epsilon^2)$ in (19).

Equations (36), (37), which define a boundary value problem for the displacements $u_{i(2)}^{(\alpha)}$ are to be complemented by the boundary conditions obtained from (23), (25) and the interface conditions (26). Since the differential equations for $u_{i(2)}^{(\alpha)}$, obtained from (36) and (37) are decoupled for each $i = 1, 2, 3$, and due to the form of the forcing terms in the boundary value problems for $u_{i(2)}^{(\alpha)}$, the solutions may be obtained in the form

$$u_{i(2)}^{(\alpha)} = P_i v_i^{(\alpha)}(x_4) + (S_{33,i} + S_{3i,3}) v_i^{*}(x_4), \quad i = 1, 2; \text{ no sum on } i, \tag{38}$$

$$u_{3(2)}^{(\alpha)} = P_3 v_3^{(\alpha)}(x_4) + S_{3j,j} w_3^{*(\alpha)}(x_4) + S_{33,3} v_3^{*(\alpha)}(x_4), \quad j = 1, 2. \tag{39}$$

The differential equations, boundary and interface conditions for $v_i^{(\alpha)}$, $v_i^{*(\alpha)}$, $w_3^{*(\alpha)}$ constitute so-called "microstructure boundary value problems" (MBVPs). These are obtained by substituting (38), (39) into (36), (37) and the appropriate interface and boundary conditions (23), (25), (26). Since the MBVPs are elementary, only the solutions are presented below:

$$v_i^{(\alpha)}(x_4) = \frac{1}{n^{(\alpha)}\mu^{(\alpha)}} q^{(\alpha)}(x_4) + \delta_{\alpha 2} \frac{n^{(1)}}{8\mu^{(1)}}, \quad i = 1, 2, \tag{40a}$$

$$v_3^{(\alpha)} = \frac{1}{n^{(\alpha)}(\lambda + 2\mu)^{(\alpha)}} q^{(\alpha)}(x_4) + \delta_{\alpha 2} \frac{n^{(1)}}{8(\lambda + 2\mu)^{(1)}}, \tag{40b}$$

$$v_i^{*(\alpha)} = -\frac{1}{n^{(\alpha)}} q^{(\alpha)}(x_4) - \delta_{\alpha 2} \frac{n^{(1)}}{8}, \quad i = 1, 2, 3, \tag{40c}$$

$$w_3^{*(\alpha)} = \frac{-\lambda^{(\alpha)}}{n^{(\alpha)}(\lambda + 2\mu)^{(\alpha)}} q^{(\alpha)}(x_4) - \delta_{\alpha 2} \frac{n^{(1)}\lambda^{(1)}}{8(\lambda + 2\mu)^{(1)}}, \tag{40d}$$

where

$$q^{(1)}(x_4) = \frac{x_4^2}{2}, \quad q^{(2)}(x_4) = \frac{|x_4|}{2} - \frac{x_4^2}{2} - \frac{n^{(1)}}{4} + \frac{n^{(1)2}}{8}. \tag{40e}$$

Equations (28), (38) and (39) furnish the displacement fields:

$$u_i^{(\alpha)} = u_{i(0)} + \epsilon S_{3i} g^{(\alpha)}(x_4) + \epsilon^2 \{P_i v_i^{(\alpha)}(x_4) + (S_{33,i} + S_{3i,3}) v_i^{*(\alpha)}(x_4)\} + 0(\epsilon^3), \quad i = 1, 2, \text{ no sum on } i; \tag{41a}$$

$$u_3^{(\alpha)} = u_{3(0)} + \epsilon S_{33} g^{(\alpha)}(x_4) + \epsilon^2 \{P_3 v_3^{(\alpha)}(x_4) + S_{3j,j} w_3^{*(\alpha)}(x_4) + S_{33,3} v_3^{*(\alpha)}(x_4)\} + O(\epsilon^3), \quad j = 1, 2. \tag{41b}$$

If eqns (41) are now averaged and $u_{i(0)}$ is subsequently eliminated, where $i = 1$–3, the constitutive relations for the interaction terms are obtained as

$$P_i = \alpha_i \left(\frac{u_i^{(2a)} - u_i^{(1a)}}{\epsilon^2} \right) + \beta_i (S_{33,i} + S_{3i,3}), \quad i = 1, 2, \text{ no sum on } i; \tag{42a}$$

$$P_3 = \alpha_3 \left(\frac{u_3^{(2a)} - u_3^{(1a)}}{\epsilon^2} \right) + \beta_3 S_{33,3} + \gamma_3 S_{3j,j}, \quad j = 1, 2. \tag{42b}$$

Here

$$\begin{aligned} \alpha_i &= 1/(v_i^{(2a)} - v_i^{(1a)}), \\ \beta_i &= (v_i^{*(1a)} - v_i^{*(2a)})\alpha_i, \quad \text{no sum on } i, \\ \gamma_3 &= (w_3^{*(1a)} - w_3^{*(2a)})\alpha. \end{aligned} \tag{43}$$

Finally, substitution of (40) into (43) furnishes

$$\begin{aligned} &\alpha_i = 12\mu_{(n)}, \quad i = 1, 2; \quad \alpha_3 = 12E_{(n)}, \\ &\beta_i = \mu_{(n)}, \quad \beta_3 = E_{(n)}, \quad \gamma_3 = G_0 E_{(n)} \end{aligned} \tag{44}$$

where

$$\mu_{(n)} = \left(\frac{n^{(1)}}{\mu^{(1)}} + \frac{n^{(2)}}{\mu^{(2)}}\right)^{-1}, \quad E_{(n)} = \left[\frac{n^{(1)}}{(\lambda+2\mu)^{(1)}} + \frac{n^{(2)}}{(\lambda+2\mu)^{(2)}}\right]^{-1},$$
$$G_0 = \frac{n^{(1)}\lambda^{(1)}}{(\lambda+2\mu)^{(1)}} + \frac{n^{(2)}\lambda^{(2)}}{(\lambda+2\mu)^{(2)}} \equiv \sum_{\alpha=1}^{2} G_0^{(\alpha p)}. \tag{45}$$

Additional constitutive relations

If the solutions of the MBVPs are used in (22b) to determine the first order stress profiles $\sigma_{ji(1)}^{(\alpha)}$, then the moment defined by (32a) can be obtained from (37)–(40); the result, with P_i related to the displacements through (42), is

$$M_{33i} = \sum_{\alpha=1}^{2} M_{33i}^{(\alpha)} = \sum_{\alpha=1}^{2} \frac{n(\alpha)}{12} P_i = \frac{P_i}{12}, \quad i = 1,2,3; \tag{46a}$$

$$M_{3ij} = \sum_{\alpha=1}^{2} M_{3ij}^{(\alpha)} = \sum_{\alpha=1}^{2} \left[\frac{1}{12}\delta_{ij}(G_1^{(\alpha p)} S_{3k,k} + G_0^{(\alpha p)} P_3) + \frac{1}{12}\mu^{(\alpha p)}(S_{3i,j} + S_{3j,i})\right] \equiv \frac{\delta_{ij}}{12}(G_1 S_{3k,k} + G_0 P_3) + \frac{1}{12}\mu_{(m)}(S_{3i,j} + S_{3j,i}), \quad i,j,k = 1,2; \tag{46b}$$

$$M_{3ij} = M_{3ji}, \quad i,j = 1,2,3. \tag{46c}$$

The quantity $G_0^{(\alpha p)}$ above is defined by (45c) and

$$G_1^{(\alpha p)} = 2\mu^{(\alpha)}\lambda^{(\alpha p)}/(\lambda+2\mu)^{(\alpha)}, \quad \mu^{(\alpha p)} = n^{(\alpha)}\mu^{(\alpha)}; \tag{46d}$$

$$G_1 = \sum_{\alpha=1}^{2} G_1^{(\alpha p)}; \quad \mu_{(m)} = \sum_{\alpha=1}^{2} \mu^{(\alpha p)}. \tag{46e}$$

It now remains to obtain a relation between R_{3i} defined by (32b) and the dependent variables. To accomplish this, the interface condition (26) with $n = 2$ will now be used. Accordingly, it is necessary to calculate $\sigma_{3i(2)}^{(\alpha)}$. The calculation is facilitated if it is observed that, on using (22b) with $n = 1$, (28), (29) and (38)–(40), eqn (22a) can be written for $n = 1$ in the form

$$\sigma_{3i(2),4}^{(\alpha)} = D_i^{(\alpha)}(x_i, t) g^{(\alpha)}(x_4). \tag{47}$$

The functions $D_i^{(\alpha)}$, which are independent of the microcoordinate, can be related to $R_{3i}^{(\alpha)}$, defined by (32b), by taking the $g^{(\alpha)}$-moment of (47); this procedure yields

$$D_i^{(\alpha)} = 12R_{3i}^{(\alpha)}/n^{(\alpha)} \tag{48}$$

whence (47) reduces to

$$\sigma_{3i(2),4}^{(\alpha)} = 12R_{3i}^{(\alpha)} g^{(\alpha)}(x_4)/n^{(\alpha)}. \tag{49}$$

The solution of (49), with the boundary condition (23) and the interface condition (26), is

$$\sigma_{3i(2)}^{(\alpha)} = 12[R_{3i}^{(\alpha)} q^{(\alpha)}(x_4)/n^{(\alpha)2} + R_{3i}^{(1)}\delta_{\alpha 2}/8] \tag{50}$$

where the functions $q^{(\alpha)}$ are given by (40e).

With the aid of (50) and the first order stress profile $\sigma_{3i(1)}^{(\alpha)}$, obtained from (37) and (38)–(40), the stress microstructure is found to be

$$\sigma_{3i}^{(\alpha)} = \sigma_{3i(0)} + \epsilon P_i g^{(\alpha)}(x_4) + 12\epsilon^2\left[\frac{R_{3i}^{(\alpha)}}{n^{(\alpha)2}} q^{(\alpha)}(x_4) + \frac{R_{3i}^{(\alpha)}}{8}\delta_{\alpha 2}\right] + O(\epsilon^3). \tag{51}$$

If (51) is now averaged for each α and if $\sigma_{3i(0)}$ is eliminated from the resulting equations, the following remarkably simple result is obtained:

$$(\sigma_{3i}^{(2\alpha)} - \sigma_{3i}^{(1\alpha)})/\epsilon^2 = R_{3i}^{(1)} + R_{3i}^{(2)} = R_{3i}. \tag{52}$$

Equation (52), with $\sigma_{3i}^{(\alpha\alpha)}$ related to the displacements through (18a), is the constitutive relation for R_{3i}.

The foregoing analysis completes the formulation of the basic differential equations for the mixture theory. The results are summarized in the next section.

SUMMARY OF MIXTURE EQUATIONS

The basic equations of the mixture theory are ($\alpha = 1, 2$; $i, j, k = 1, 2, 3$ unless noted otherwise):

(a) *Equations of motion*:

$$\sigma_{ji,j}^{(\alpha\beta)} - \rho^{(\alpha\beta)}\ddot{u}_i^{(\alpha\alpha)} = (-1)^{\alpha}P_i, \quad \sigma_{ij}^{(\alpha\beta)} = \sigma_{ji}^{(\alpha\beta)}; \tag{15}$$

$$M_{3ji,j} - \frac{\rho_{(m)}}{12}\ddot{S}_{3i} = -R_{3i}, \quad M_{3ji} = M_{3ij}; \tag{31}$$

(b) *Constitutive equations*:

$$\sigma_{ij}^{(\alpha\beta)} = [\lambda^{(\alpha\beta)}u_{k,k}^{(\alpha\alpha)}\delta_{ij} + \mu^{(\alpha\beta)}(u_{i,j}^{(\alpha\alpha)} + u_{j,i}^{(\alpha\alpha)})] + (-1)^{\alpha+1}[\lambda^{(\alpha)}S_{33}\delta_{ij} + \mu^{(\alpha)}(\delta_{j3}S_{3i} + \delta_{i3}S_{3j})]; \tag{18a}$$

$$M_{3ji} = \frac{\delta_{ij}}{12}(G_1 S_{3k,k} + G_0 P_3) + \frac{\mu_{(m)}}{12}(S_{3i,j} + S_{3j,i}), \qquad i, j, k = 1, 2; \tag{46b}$$

$$M_{33i} = P_i/12, \quad i = 1, 2, 3; \tag{46a}$$

$$P_i = \alpha_i \frac{(u_i^{(2\alpha)} - u_i^{(1\alpha)})}{\epsilon^2} + \beta_i(S_{33,i} + S_{3i,3}), \quad i = 1, 2, \text{ no sum on } i; \tag{42a}$$

$$P_3 = \alpha_3 \frac{(u_3^{(2\alpha)} - u_3^{(1\alpha)})}{\epsilon^2} + \beta_3 S_{33,3} + \gamma_3 S_{3j,j}; \quad j = 1, 2; \tag{42b}$$

$$R_{3i} = (\sigma_{3i}^{(2\alpha)} - \sigma_{3i}^{(1\alpha)})/\epsilon^2. \tag{52}$$

Here

$$G_0 = \sum_{\alpha=1}^{2} \lambda^{(\alpha\beta)}/(\lambda + 2\mu)^{(\alpha)}, \quad G_1 = 2\sum_{\alpha=1}^{2} \lambda^{(\alpha\beta)}\mu^{(\alpha)}/(\lambda + 2\mu)^{(\alpha)}; \tag{45c, 46d}$$

$$\alpha_i = 12\mu_{(n)}, \quad \beta_i = \mu_{(n)}, \quad i = 1, 2, \qquad \alpha_3 = 12E_{(n)}, \quad \beta_3 = E_{(n)}, \quad \gamma_3 = G_0 E_{(n)}; \tag{44}$$

$$\mu_{(n)}^{-1} = \sum_{\alpha=1}^{2} n^{(\alpha)}/\mu^{(\alpha)}, \quad E_{(n)}^{-1} = \sum_{\alpha=1}^{2} n^{(\alpha)}/(\lambda + 2\mu)^{(\alpha)}, \tag{45}$$

$$\lambda^{(\alpha\beta)} = n^{(\alpha)}\lambda^{(\alpha)}, \quad \mu^{(\alpha\beta)} = n^{(\alpha)}\mu^{(\alpha)}, \quad \mu_{(m)} = \sum_{\alpha=1}^{2} \mu^{(\alpha\beta)}. \tag{18b, 46e}$$

If the stress-type variables are eliminated by use of the appropriate constitutive equations, the above set can be reduced to nine differential equations with *constant* coefficients in the nine unknowns $u_i^{(\alpha\alpha)}$, S_{3i}, $i = 1, 2, 3$, $\alpha = 1, 2$. All coefficients in these equations are determined once the elastic properties of the constituents and their volume fractions are known.

VARIATIONAL PRINCIPLE AND BOUNDARY CONDITIONS

An important question that has not yet been addressed is that of appropriate initial and boundary data for the solution of the mixture equations. This question can evidently be

completely resolved only by a boundary layer analysis wherein the edge-zone solutions are matched with the mixture solutions. However, the need for a boundary layer analysis can be alleviated to some extent by use of a variational approach to determine the appropriate boundary data for the mixture differential equations. Such an analysis, which is similar to that used for plate and shell theories in elasticity, is presented in this section.

For purposes of constructing a variational principle, it will be convenient to introduce the following matrix notation:

$$\mathbf{e}^{(\alpha)T} = [e_{11}, e_{22}, e_{33}, 2e_{23}, 2e_{31}, 2e_{12}]^{(\alpha)}, \tag{53}$$

where

$$e_{ij}^{(\alpha)} \equiv \frac{1}{2}(u_{i,j}^{(\alpha a)} + u_{j,i}^{(\alpha a)}); \tag{54}$$

$$\boldsymbol{\kappa}^T = [\kappa_{311}, \kappa_{322}, \kappa_{333}, 2\kappa_{323}, 2\kappa_{331}, 2\kappa_{312}], \tag{55}$$

where

$$\kappa_{3ij} \equiv \frac{1}{2}(S_{3i,j} + S_{3j,i}); \tag{56}$$

$$\mathbf{K}^{(\alpha)} = [K_{ij}^{(\alpha)}], \quad K_{ij}^{(\alpha)} = \begin{cases} (\lambda + 2\mu)^{(\alpha\rho)} & \text{if } i = j = 1,2,3, \\ \lambda^{(\alpha\rho)} & \text{if } i \neq j \text{ and } 1 \leq i,j \leq 3, \\ \mu^{(\alpha\rho)} & \text{if } i = j = 4,5,6, \\ 0 & \text{otherwise;} \end{cases} \tag{57}$$

$$\mathbf{L} = [L_{ij}], \quad \frac{12}{\epsilon^2} L_{ij} = \begin{cases} G_1 + G_0^2 E_{(n)} + 2\mu_{(m)} & \text{if } i = j = 1,2, \\ E_{(n)} & \text{if } i = j = 3, \\ G_0 E_{(n)} & \text{if } i \neq j \text{ and } i \text{ or } j = 3, \\ \mu_{(m)} & \text{if } i = j = 4,5,6, \\ G_1 + G_0^2 E_{(n)}, & i \neq j \text{ and } i,j = 1,2, \end{cases} \tag{58}$$

Equations (57) and (58) define 6×6 matrices of material constants while eqns (53) and (55) represent the macro- and microstrain components, respectively.

Energy density-type quantities are now defined according to ($\alpha = 1$ or 2)

$$U^{(\alpha\alpha)} = \frac{1}{2}\mathbf{e}^{(\alpha)T^{(\alpha)}}\mathbf{K}\mathbf{e}^{(\alpha)} + \sum_{i=1}^{2}(-1)^{\alpha+1}S_{3i}[(\mu_{(n)} + \mu^{(\alpha)})u_{i,3}^{(\alpha a)}$$
$$+ (G_0 E_{(n)} + \mu^{(\alpha)})u_{3,i}^{(\alpha a)}] + (-1)^{\alpha+1}S_{33}[(\mu_{(n)} + \lambda^{(\alpha)})(e_{11}^{(\alpha)} + e_{22}^{(\alpha)}) - (E_{(n)} + [\lambda + 2\mu]^{(\alpha)}e_{33}^{(\alpha)}],$$

$$2U^{(12)} = \sum_{i=1}^{2}\left[\frac{\alpha_i}{\epsilon^2}(u_i^{(1a)} - u_i^{(2a)})^2\right] + \sum_{\alpha=1}^{2}\left[\frac{\mu^{(\alpha)}}{n^{(\alpha)}}(S_{31}^2 + S_{32}^2) + \frac{\lambda^{(\alpha)} + 2\mu^{(\alpha)}}{n^{(\alpha)}}S_{33}^2\right] + \boldsymbol{\kappa}^T\mathbf{L}\boldsymbol{\kappa}.$$

With the aid of the foregoing notation, the functional

$$\Pi = \int\int_D\int [U^{(11)} + U^{(22)} + U^{(12)}]\,dx_1\,dx_2\,dx_3 - \oint\!\!\!\oint_{\partial D_S}\left[\sum_{\alpha=1}^{2} u_i^{(\alpha a)}\bar{T}_i^{(\alpha)} + S_{3i}\bar{M}_{3i}\right]d\sigma \tag{60}$$

is introduced, where ∂D_S is the part of the boundary of D on which the traction vector-type quantities $\bar{T}_i^{(\alpha)}$ and $\bar{M}_{3i}$ are prescribed, and $d\sigma$ denotes the elemental area on the boundary. The precise relationship between $\bar{T}_i^{(\alpha)}$ and $\bar{M}_{3i}$ on the one hand and the stress components $\sigma_{ij}^{(\alpha)}$ and M_{3ij} on the other, is obtained by the requirement that, with (53)–(59) as definitions, Π be an extremum with respect to $u_i^{(\alpha a)}$ and S_{3i}. This leads to the differential equations of the mixture theory in D, and appropriate boundary conditions on ∂D, all as the Euler equations of the variational problem.

Following substitution of the various definitions in (60), performing the first variation of Π with respect to $u_i^{(\alpha\alpha)}$ and S_{3i}, and using Green's theorem, one obtains

$$\delta\Pi = -\int\int_D \int \sum_{i=1}^{3} \left[\sum_{\alpha=1}^{2} \delta u_i^{(\alpha\alpha)}(\,) + \delta S_{3i}(\,) \right] dx_1\, dx_2\, dx_3 + \oiint_{\partial D_S} \left\{ \sum_{\alpha=1}^{2} (\sigma_{ji}^{\prime(\alpha p)}\nu_j - \bar{T}_i^{(\alpha)})\delta u_i^{(\alpha\alpha)} + (M'_{3ji}\nu_j - \bar{M}_{3i})\delta S_{3i} \right\} d\sigma \tag{61}$$

where the coefficients of $\delta u_i^{(\alpha\alpha)}$ and δS_{3i} in the volume integral are precisely the differential equations of the mixture theory without the inertia terms. In the surface integral of (61), ν_i denotes the components of the outward normal to the surface, and the modified stress components are defined as

$$[\sigma_{ij}^{\prime(\alpha p)}]^T = [\sigma_{ij}^{(\alpha p)}]^T + (-1)^{\alpha+1} \begin{bmatrix} \mu_{(n)}S_{33} & 0 & \mu_{(n)}S_{31} \\ 0 & \mu_{(n)}S_{33} & \mu_{(n)}S_{32} \\ G_0E_{(n)}S_{31} & G_0E_{(n)}S_{32} & E_{(n)}S_{33} \end{bmatrix}, \tag{62}$$

$$\mathbf{m}' = \mathbf{L}\boldsymbol{\kappa}, \tag{63}$$

where

$$m'^T = [M'_{311}, M'_{322}, M'_{333}, M'_{323}, M'_{331}, M'_{312}]. \tag{64}$$

It thus follows from (61) that the prescribed traction-type quantities are of the form $\sigma_{ji}^{\prime(\alpha p)}\nu_j$ and $M'_{3ji}\nu_j$. Thus, based on the foregoing analysis, it is concluded that on ∂D one must prescribe the quantities

$$\begin{aligned} &u_i^{(\alpha\alpha)} \quad \text{or} \quad \sigma_{ji}^{\prime(\alpha p)}\nu_j, \\ &S_{3i} \quad \text{or} \quad M'_{3ji}\nu_j. \end{aligned} \tag{65}$$

It is of interest to note that the modified partial stress tensor $\sigma_{ij}^{\prime(\alpha p)}$ is not symmetric. This result is consistent with the theory of elasticity with couple stresses.

With regard to the question of appropriate initial data to be used with the mixture theory, one could appeal to Hamilton's principle. This, however, is not necessary since only the second derivative with respect to time appears in the mixture equations, as in the theories for homogeneous media. Consequently, it is appropriate to prescribe

$$u_i^{(\alpha\alpha)},\ S_{3i},\ \dot{u}_i^{(\alpha\alpha)},\ \dot{S}_{3i} \quad \text{at} \quad t = 0 \quad \text{in} \quad D. \tag{66}$$

RECOVERY OF MICROSTRUCTURE

Formulation of the mixture theory is now complete. The macroresponse of a laminated medium is obtained by solving the differential equations of the mixture theory together with appropriate initial boundary data consistent with (65) and (66). Although the solution furnishes only averages of the field quantities, the fields $u_i^{(\alpha\alpha)}(x_k, t)$, $S_{3i}(x_k, t)$ can be used to recover the microstructure by the following procedure. First, by averaging (41a) and (41b) one obtains

$$u_{i(0)} = u_i^{(\alpha\alpha)} - \epsilon^2[P_i v_i^{(\alpha\alpha)} + (S_{33,i} + S_{3i,3})v_i^{*(\alpha\alpha)}], \quad i = 1, 2; \text{ no sum on } i; \tag{67a}$$

$$u_{3(0)} = u_3^{(\alpha\alpha)} - \epsilon^2[P_3 v_3^{*(\alpha\alpha)} + S_{3j,j}w^{*(\alpha\alpha)} + S_{33,3}v_3^{*(\alpha\alpha)}], \quad j = 1,2. \tag{67b}$$

Next, the quantity P_i is obtained from (42). Finally, $u_{i(0)}$, obtained from (67), is used to compute the displacement microstructure, correct to $O(\epsilon^2)$, from (41).

To determine the stress microstructure, one first obtains from (31, 32),

$$R_{3i}^{(\alpha)} = (\rho^{(\alpha p)}/12)\ddot{S}_{3i} - M_{3ji,j}^{(\alpha)} \tag{68}$$

where $M_{3\beta}^{(\alpha)}$ is obtained from (46b). Further, the average of (51) yields

$$\sigma_{3i(0)} = \sigma_{3i}^{(\alpha a)} - 12\epsilon^2\left[\frac{R_{3i}^{(\alpha)}}{n^{(\alpha)2}} q^{(2a)} + \frac{R_{3i}^{(1)}}{8}\delta_{\alpha 2}\right]. \tag{69}$$

Finally, the stress microstructure, correct to $O(\epsilon^2)$, is calculated from (51).

EXACT VS MIXTURE PHASE VELOCITY SPECTRA

Propagation at olbique incidence in plane strain

In an attempt to test the efficacy of the mixture model, the phase velocity spectra of the mixture theory have been compared with the exact results obtained by Sve[18] for time harmonic waves propagating at an arbitrary angle of incidence in the plane normal to the material interfaces. Such waves are obtained for the case of plane strain ($u_2^{(\alpha a)} = 0$, $S_{32} = 0$) by assuming motion of the form

$$\begin{Bmatrix} u_1^{(\alpha a)} \\ u_3^{(\alpha a)} \\ S_{31} \\ S_{33} \end{Bmatrix} = \begin{Bmatrix} U_1^{(\alpha)} \\ U_3^{(\alpha)} \\ iks_1 \\ iks_3 \end{Bmatrix} \exp\left[ik(x_1 \cos\theta + x_3 \sin\theta) - i\omega t\right]. \tag{70}$$

Substitution of (70) into the mixture eqns (15), (31) written in terms of displacements by use of the constitutive relations, leads to the eigenvalue problem

$$\mathbf{QU} = (\epsilon\omega)^2\mathbf{U} \tag{71}$$

where Q is a 6×6 matrix, the elements of which are functions of the mixture constants and the wave number, ϵk. The vector U in (71) is given by

$$\mathbf{U}^T = [U_1^{(1)}, U_1^{(2)}, U_3^{(1)}, U_3^{(2)}, s_1, s_2]. \tag{72}$$

Following calculation of the eigenvalues ($\epsilon\omega$) as a function of (ϵk) from (71), the phase velocity is obtained from

$$C_p = (\epsilon\omega)/(\epsilon k). \tag{73}$$

The dispersion curves obtained from the mixture theory are shown in Figs. 3(a)–(f) wherein spectra from the well-known effective stiffness theory[1], which has a comparable number of

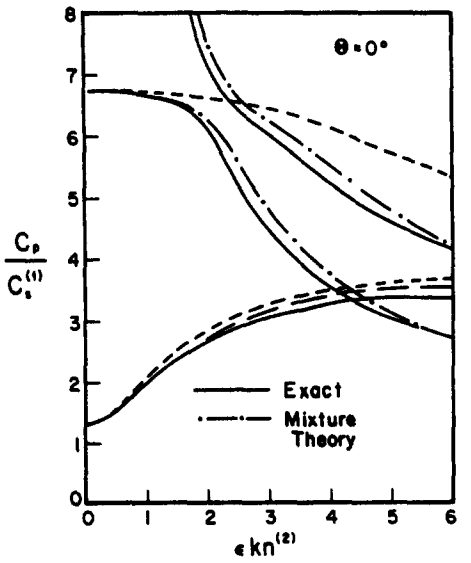

Fig. 3(a). Phase velocity spectra for $\theta = 0°$. (—— exact solution; -·-· mixture theory; ---- effective stiffness theory.)

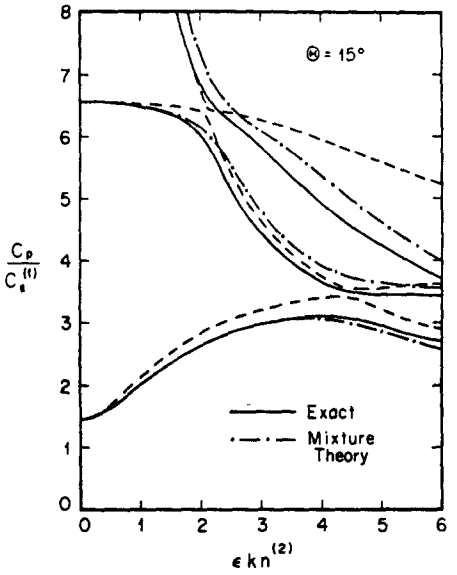

Fig. 3(b). Phase velocity spectra for $\theta = 15°$.

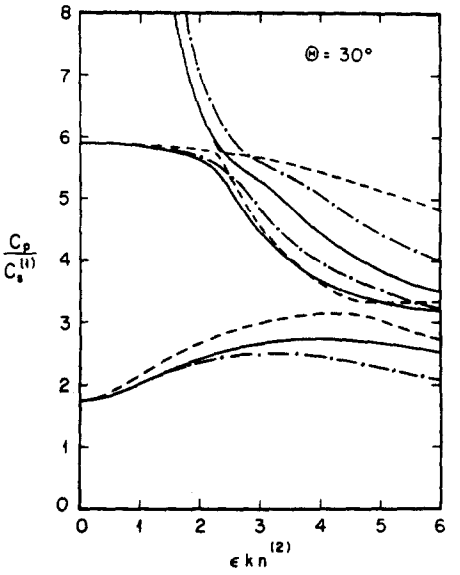

Fig. 3(c). Phase velocity spectra for $\theta = 30°$.

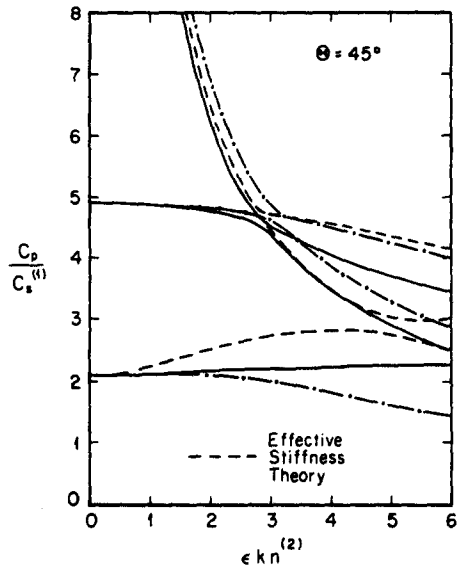

Fig. 3(d). Phase velocity spectra for $\theta = 45°$.

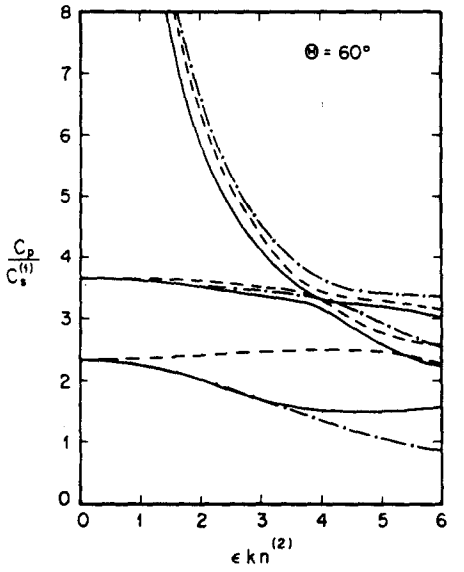

Fig. 3(e). Phase velocity spectra for $\theta = 60°$.

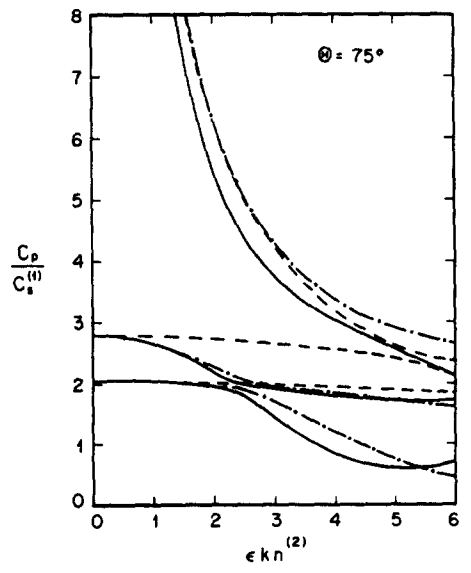

Fig. 3(f). Phase velocity spectra for $\theta = 75°$.

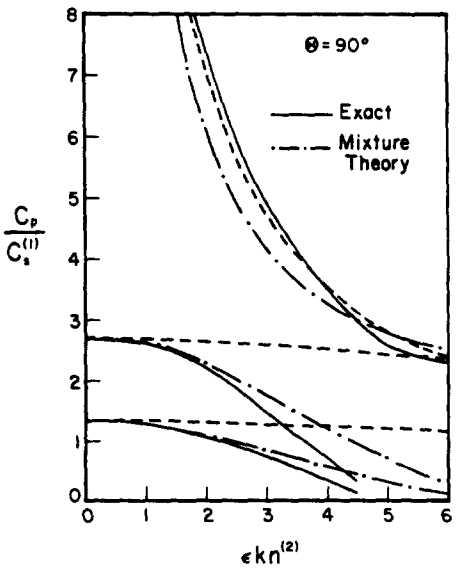

Fig. 3(g). Phase velocity spectra for $\theta = 90°$.

unknowns, are also shown for comparison of accuracy. The material properties used for the computation are as follows:

$$\mu^{(2)}/\mu^{(1)} = 50, \quad \lambda^{(1)}/\mu^{(1)} = 2.33, \quad \lambda^{(2)}/\mu^{(1)} = 75,$$
$$\rho^{(2)}/\rho^{(1)} = 3, \quad n^{(2)} = 0.8. \tag{74}$$

Figures 3(a)–(g) exhibit two acoustic modes and an optical mode, the physical interpretation of which can be found in [1]. In Figs. 3(a)–(g) phase velocities are nondimensionalized by the shear wave velocity of material 1: $C_s^{(1)} = \mu^{(1)}/\rho^{(1)}$. As is evident from the comparisons, the mixture theory is, in general, more accurate than the first order effective stiffness theory. In order to demonstrate the capability of the mixture theory to model the banded and periodic structure on the frequency-wave number plane for waves propagating perpendicular to the layering, another view of Fig. 3(g) is furnished in Fig. 4 where $\Omega = \epsilon\omega n^{(2)}$. This figure shows that the mixture theory can model the pass bands and stop bands predicted by the elasticity solution with reasonable accuracy.

Based on Figs. 3(a)–(g) it is concluded that the mixture theory is more accurate than the *first order* effective stiffness theory. However, a higher order effective stiffness theory is capable of predicting, with increased accuracy, this dispersion phenomena and a comparison of the latter with the mixture theory is in order. In Figs. 5(a)–(c) phase velocity spectra have been calculated by the mixture theory for the case used to exhibit the second-order effective stiffness theory[21]:

$$\mu^{(2)}/\mu^{(1)} = 100, \quad \lambda^{(1)}/2(\lambda^{(1)} + \mu^{(1)}) = 0.35,$$
$$\lambda^{(2)}/2(\lambda^{(2)} + \mu^{(2)}) = 0.3, \quad \rho^{(2)}/\rho^{(1)} = 3,$$
$$n^{(2)} = 0.8. \tag{75}$$

It can be concluded from Figs. 5(a)–(c) that the mixture theory can predict the dispersion phenomena as well as the second order effective stiffness theory which, it is noted, requires 12 equations of motion and 8 constraint conditions while the mixture theory only requires 6 equations of motion.

The dispersion curves, Figs. 3(a)–(g), 5(a)–(c) are representative of global or macroscopic motion. As an example of the ability of the mixture theory to predict details of the stress and displacement profiles, the stress profiles for sinusoidal waves propagating normal to the layers ($\theta = 90°$ in (70)), as predicted by the mixture theory, have been compared with the exact solution given by Lee[22]. Prior to discussion of this comparison, however, it is noted that, if the wavelength of the propagating wave is used for nondimensionalization, then the scaled

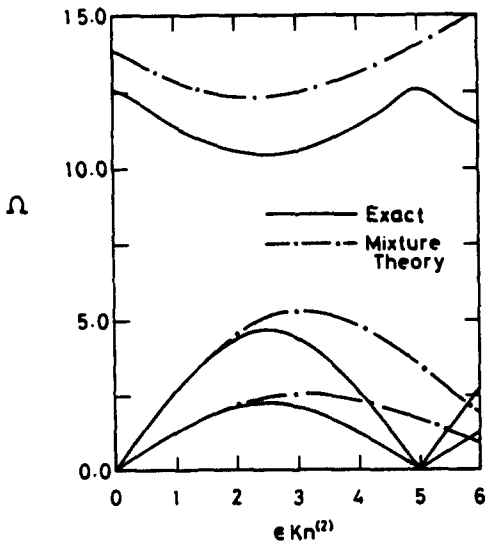

Fig. 4. Frequency vs wave number for normal incidence.

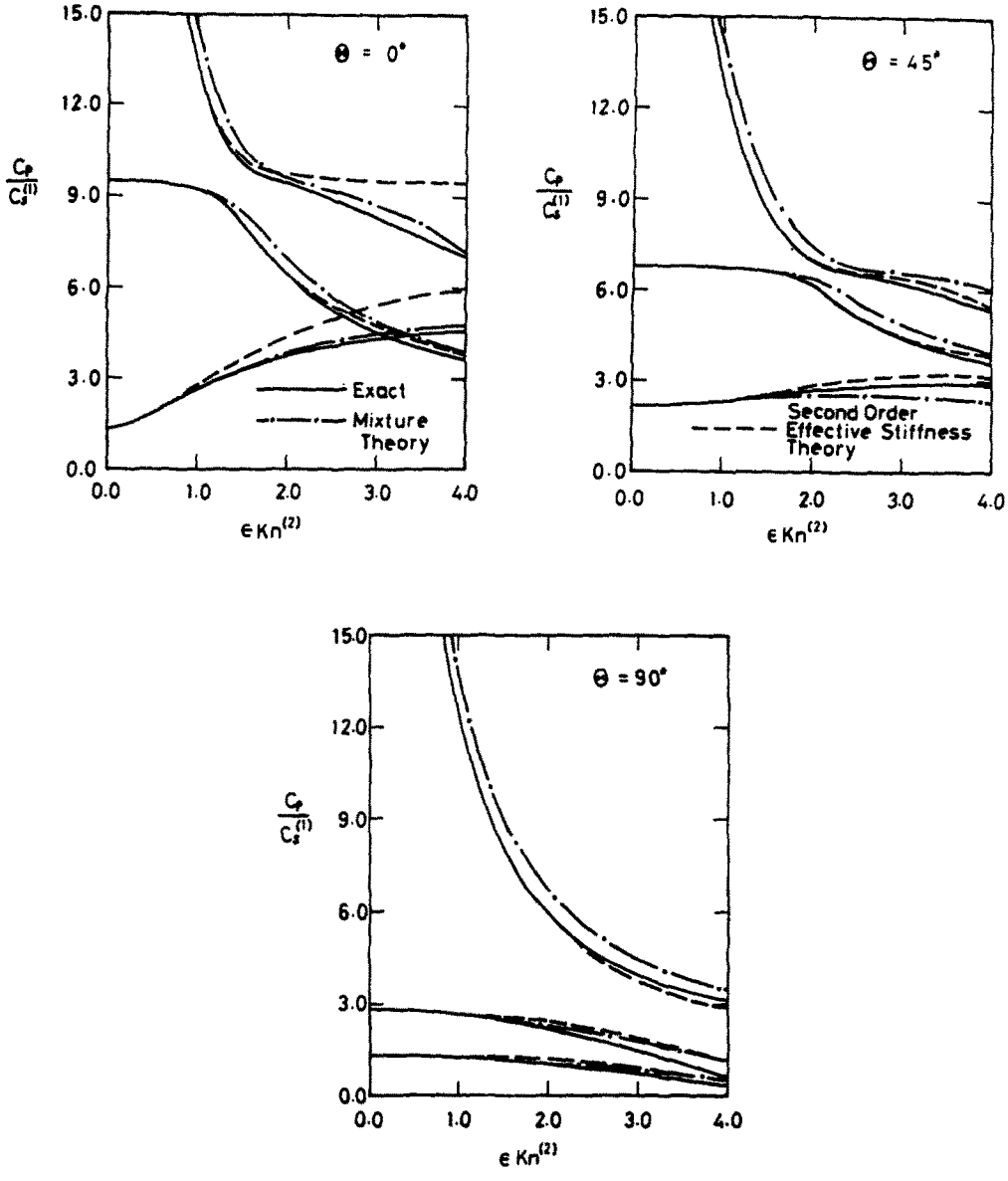

Fig. 5(a). Phase velocity spectra for $\theta = 0°$. (—— exact solution; -·-· mixture theory; ---- second order effective stiffness theory.)

Fig. 5(b). Phase velocity spectra for $\theta = 45°$.

Fig. 5(c). Phase velocity spectra for $\theta = 90°$.

wave number is 2π and therefore $\epsilon = \epsilon k/2\pi$. It is also noted[22] that the stress distribution in the whole space can be obtained from the distribution within half of the unit cell—a result which is a consequence of the quasi-periodicity of the Floquet waves and the antisymmetry of the stress profile within the unit cell.

The microstructure stress profile comparisons, which are shown in Fig. 6, have been made for two values of ϵ and two values of the ratio of constituent moduli $E^{(\alpha)} = \lambda^{(\alpha)} + 2\mu^{(\alpha)}$ or $\mu^{(\alpha)}$ corresponding to longitudinal wave propagation or shear wave propagation. For $\epsilon = 1/8$, the mixture stress profile is almost identical to the exact solution. For $\epsilon = 7/16$, the mixture predictions are still quite accurate, even though the ratio of microdimension to signal wavelength is close to 1/2. It should be added here that the mixture theory predicts a continuous stress profile. This feature is absent from the effective stiffness theories; corresponding results were therefore not presented in Fig. 6.

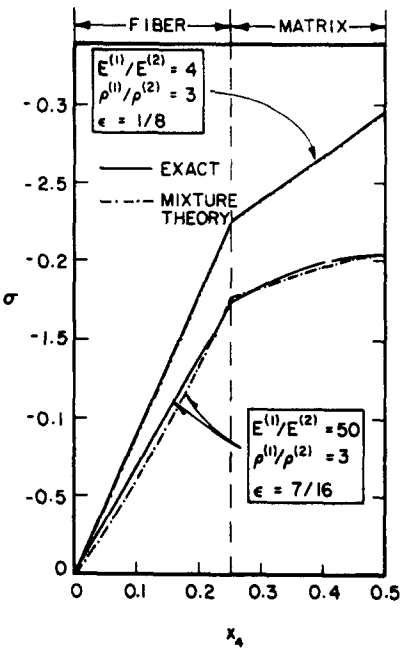

Fig. 6. Stress microstructure for normal incidence.

The variation of stress profile in a given spatial domain as the wave passes through that domain is illustrated in Figs. 7(a), (b) for two values of time. Since these results are based on Fig. 6, they exhibit the same degree of accuracy. The exact elasticity solution has been obtained in [22]. The stress profile of the mixture theory is computed by obtaining the eigenvector in (71) for $u_3^{(\alpha\alpha)}$ and S_{33} and using the results of (68), (69) and (51) to recover the stress microstructure.

CONCLUDING REMARKS

The asymptotic method of multiple scales has been used to construct a continuum theory for the elastodynamic behavior of periodically laminated media consisting of two constituents. The resulting theory was cast in the form of a binary mixture theory. Comparisons of exact vs mixture phase velocity spectra imply that good accuracy may be obtained via the mixture theory for problems where the energy is partitioned primarily into the first two acoustic modes

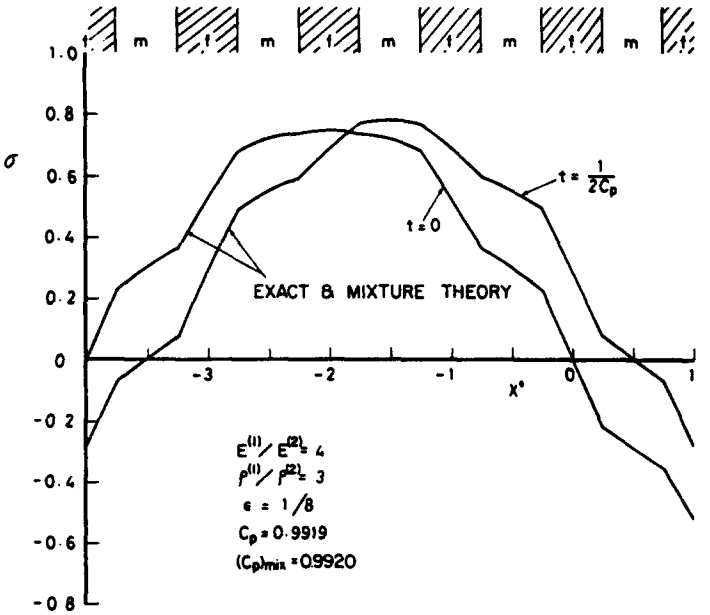

Fig. 7(a). Stress profile for normal incidence.

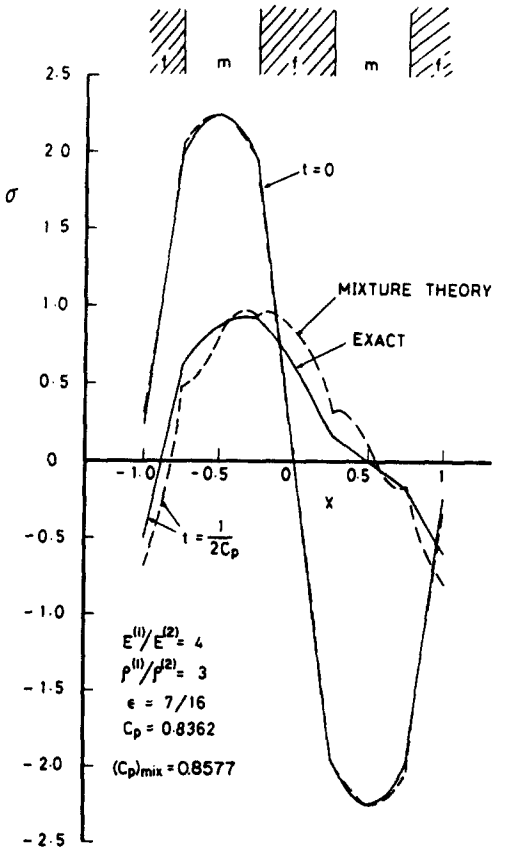

Fig. 7(b). Stress profile for normal incidence.

and where the material microdimension does not exceed one-half the dominant signal wavelength. For such cases, the theory appears to be an attractive alternative to direct numerical solution of initial boundary value problems for laminated composites.

It is appropriate to conclude this paper with an elaboration of similarities and the differences between the approach described herein and the method of homogenization for periodic composites which has attracted much attention in the recent literature[16, 17, 23–25]. Two essential aspects of both the methods are the introduction of the microcoordinate and the assumption of periodicity with respect to this coordinate. A consequence of these features of the techniques is that no explicit smoothing operation is required, in contrast to the previous treatments of the laminate problem[1, 2]. However, the main emphasis in this paper has been on the development of a dispersive theory for dynamic problems, whereas in most other studies[16, 17, 23, 24] the major objective appears to be the asymptotic justification of classifical effective modulus models. Although it is not necessary to resort to a mixture theoretic formalism for construction of higher order models, and a direct asymptotic approach can easily be utilized in the manner of [25], such theories typically contain the spatial and temporal derivatives of the field quantities which are of higher order than second order derivatives that occur in the mixture theory. Consequently, questions about the specification of appropriate initial and boundary data in the theories based on the direct approach are more difficult to resolve than they are in the mixture formulation. An additional advantage of the approach described in this paper is, of course, the fact that the derived field equations can be interpreted from a physical point of view in a rather convenient fashion.

REFERENCES

1. C. T. Sun, J. D. Achenbach and G. Herrmann, Continuum theory for a laminated medium. *J. Appl. Mech.* **35**, 467 (1968).
2. G. A. Hegemier and T. C. Bache, A general continuum theory with microstructure for wave propagation in elastic laminated composites. *J. Appl. Mech.* **41**, 101 (1974).

3. J. D. Cole, *Perturbation Methods in Applied Mathematics*. Blaisdell, Waltham, Mass. (1968).
4. G. A. Hegemier, Finite amplitude elastic-plastic wave propagation in laminated composites. *J. Appl. Phys.* **45**, 4248 (1974).
5. G. A. Hegemier, Mixture theories with microstructure for wave propagation and diffusion in composite materials. In *Proc. 2nd Int. Symp. on Continuum Models of Discrete Systems* (Edited by J. Provan). Montreal (1977).
6. H. Murakami, A. Maewal and G. A. Hegemier, Mixture theory for propagation of longitudinal waves in unidirectional composites with fibers of arbitrary cross-section—I: Formulation. *Int. J. Solids Structures* **15**, 325 (1979).
7. E. Cosserat and F. Cosserat, *Theorie des Corps déformables*. Hermann, Paris (1909).
8. J. L. Ericksen and C. Truesdell, Exact theory of stress and strain in rods and shells. *Arch. Rat. Mech. Anal.* **16**, 51 (1964).
9. A. E. Green and P. M. Naghdi, On basic equations for mixtures. *Quart. J. Mech. Appl. Math.* **9**, 329 (1970).
10. R. M. Bowen, Theory of mixtures. In *Continuum Physics—III* (Edited by A. C. Eringen). Academic Press, New York (1976).
11. S. J. Allen and K. A. Kline, A mixture theory with microstructures. *ZAMP* **20**, 145 (1969).
12. R. J. Twiss and A. C. Eringen, Theory of mixtures for micromorphic materials—1, Balance laws. *Int. J. Engng Sci.* **9**, 1019 (1971).
13. A. Bedford and M. Stern, Toward a diffusing continuum theory of composite materials. *J. Appl. Mech.* **38**, 8 (1971).
14. G. A. Hegemier, G. A. Gurtman and A. H. Nayfeh, A continuum mixture theory of wave propagation in layered and fiber reinforced composites. *Int. J. Solids Structures* **9**, 395 (1973).
15. H. Murakami, On mixture theories for diffusion and wave propagation in composite materials. Ph.D. Dissertation. University of California, San Diego (1978).
16. J. M. Burgers, On some problems of homogenization. *Quart. Appl. Math.* **XXXV**, 421 (1978).
17. I. Babuska, Solution of interface problems by homogenization—I. *SIAM J. Math. Anal.* **I**, 603 (1976).
18. C. Sve, Time harmonic waves travelling obliquely in a periodically laminated medium. *J. Appl. Mech.* **38**, 477 (1971).
19. W. Kohn, Propagation of low frequency elastic disturbances in a composite material. *J. Appl. Mech.* **41**, 97 (1974).
20. L. Brillouin, *Wave Propagation in Periodic Structures*. Dover (1953).
21. D. S. Drumheller and A. Bedford, Wave propagation in elastic laminates using a second order microstructure theory. *Int. J. Solids Structures* **10**, 61 (1974).
22. E. H. Lee, A survey of variational methods for elastic wave propagation analysis in composites with periodic structures. *Dynamics of Composite Materials*, p. 122. ASME (1972).
23. J. B. Keller, Effective behavior of heterogeneous media. *Statist. Mechan. and Statist. Meth. in Theory and Appl.* (Edited by U. Landman), p. 631. Plenum Press, New York (1977).
24. A. Bensoussan, J. L. Lions and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*. North-Holland (1978).
25. A. Maewal, Homogenization for transient heat conduction. *J. Appl. Mech.* **46**, 945 (1979).